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The primary focus of the project was on revealing foundational role of underlying dynamical structure and sparse information structure of large-scale dynamical networks in emergence of severe theoretical fundamental limits on the resulting performance and robustness in such networks. Our research results provide a novel unified approach for analysis and synthesis of large-scale dynamical network. We have developed an axiomatic general theory of systemic measures for performance and robustness analysis and synthesis of large-scale dynamical networks.

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Abstract

The goal of this research is the development of new fundamental insights and methodologies to exploit structural properties of large-scale dynamical networks in order to quantify their inherent *fundamental limits* on performance and robustness and characterize their resulting *fundamental tradeoffs*. The primary focus of the project was on revealing foundational role of underlying *dynamical structure* and *sparse* information structure of large-scale dynamical networks in emergence of severe theoretical fundamental limits on the resulting performance and robustness in such networks. Our research results provide a novel unified approach for analysis and synthesis of large-scale dynamical network. We have developed an axiomatic general theory of *systemic measures* for performance and robustness analysis and synthesis of large-scale dynamical networks. The proposed theory evolves around introducing the new classes of convex and Schur-convex systemic measures that capture transient, steady-state, macroscopic, and microscopic features of dynamical networks. The notion of systemic measure provides a unifying umbrella under which an integrated theory of hard limits based on operator, graph, optimization theories can be developed to systematically deal with uncertainty, performance, and robustness in distributed control and dynamical systems. Our research results are highly relevant for analysis and synthesis of engineered and natural dynamical networks. In the following pages, I will briefly explain our theoretical achievements. More details have been provided for selected number of topics. Proper citations to our published papers have been provided for further details and readings.

Emergence of Fundamental Limits in Spatially Distributed Dynamical Networks and Their Tradeoffs

Summary of Accomplishments and Research Results

1 Systemic Performance and Robustness Measures

One of the simplest class of dynamical networks that our proposed methodology can be explained in a simple setting is the class of first–order linear consensus networks with the following dynamics

$$\dot{x}(t) = -L_{\mathcal{G}}\, x(t) + \xi(t) \tag{1}$$

$$y(t) = x(t) - \bar{x}(t)\mathbf{1}_n \tag{2}$$

where $x(t), y(t) \in \mathbb{R}^n$ are the state variable and output of the dynamical network, respectively. The $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n$ and $\bar{x}(t)$ is the average of all state variables $x_1(t), \ldots, x_n(t)$. The exogenous disturbance input $\xi(t)$ captures the uncertain effect of the environment on the dynamics of the network, which is assumed to be a zero–mean Gaussian white noise with unit covariance. This dynamical network is defined over a graph $\mathcal{G} = \big(V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G})}\big)$, where $V(\mathcal{G})$ is the set of nodes and $E(\mathcal{G})$ is the set of edges. The weight function $w^{(\mathcal{G})}$ is a map that is defined from the set of edges to the set of strictly positive real numbers and it captures the coupling structure of the underlying dynamical network. The Laplacian matrix of the underlying graph of the network is denoted by $L_{\mathcal{G}}$. The Laplacian matrix is a positive semidefinite matrix and can be used to find many properties of the corresponding graph. The (i, i) diagonal element of the Laplacian matrix is equal to the sum of the weights associated to every edge incident to node i and its (i, j) off–diagonal element is equal to weight of edge connecting nodes i and j, otherwise it is equal to zero. In the following discussions, we assume that all underlying graphs are finite, simple, undirected and connected, and therefore, the smallest Laplacian eigenvalue is always equal to zero. Therefore, the dynamical network (1)-(2) is marginally stable. However, this network is well-defined and its marginally stable mode is not observable from the output.

The linear dynamical network (1)-(2) can be viewed as a system that has been already stabilized by a linear state feedback control law and operating in closed-loop. The sparsity pattern of the Laplacian matrix $L_{\mathcal{G}}$ is imposed by the topology of the underlying graph $\mathcal{G}$ and the corresponding weight function, which models the coupling structure and strength among the subsystems in the closed-loop system. The existence of such inherent sparsity-constraints on the topology of the underlying graphs play a foundational role in emergence of severe theoretical fundamental limits on the global performance and robustness of this class of dynamical networks. We use the following simple but illustrative example to conceptualize the idea.

Example: Let us index the Laplacian eigenvalues in ascending order $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, where the smallest eigenvalue λ_1 is always equal to zero for connected graphs. The steady state variance of the output of the network (1)-(2) is an example of a performance and robustness measure. This measure quantifies the expected dispersion of the state of the network in state steady [26] and is indeed equal to the $\mathcal{H}_2$–norm of the system from the disturbance input to the output of the network and can be calculated in terms of Laplacian eigenvalues as follows

$$\rho_{\mathcal{H}_2}\big(\mathcal{N}(L_{\mathcal{G}})\big) = \sqrt{\lim_{t \to \infty} \mathbb{E}\big[y(t)^T y(t)\big]} = \frac{1}{\sqrt{2}} \left[\sum_{i=2}^{n} \lambda_i^{-1} \right]^{\frac{1}{2}}, \tag{3}$$
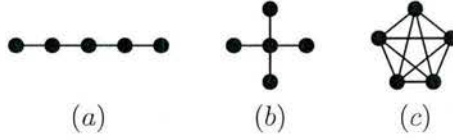
Figure 1: Among all connected graphs with five nodes, the value of the systemic measure (3) is (a) maximal for a path graph among all graphs as well as among all tree graphs, (b) minimal for star graph among all tree graphs, and (c) minimal for a complete graph among all graphs.

where $\mathbb{E}[.]$ is the expected value operator. Figure 1 shows a simple but illustrative example that how measure (3) depends on the topology of the underlying graph of networks. In our results, we take an axiomatic approach to classify several important performance and robustness measures for linear consensus networks. Let us represent the set of all linear consensus networks in the form (1)-(2) that have a common initial condition x_0 by $\mathfrak{N}(x_0)$. Therefore, every dynamical network in $\mathfrak{N}(x_0)$ can be represented using its Laplacian matrix using symbol $\mathcal{N}(L_{\mathcal{G}})$.

Definition 1 *For every given $\mathcal{N}(L_{\mathcal{G}_1}), \mathcal{N}(L_{\mathcal{G}_2}) \in \mathfrak{N}(x_0)$, the addition and scalar multiplication operations on $\mathfrak{N}(x_0)$ is defined as follows:*

(i) $\mathcal{N}(L_{\mathcal{G}_1}) + \mathcal{N}(L_{\mathcal{G}_2}) = \mathcal{N}(L_{\mathcal{G}_1} + L_{\mathcal{G}_2})$

(ii) $\alpha \mathcal{N}(L_{\mathcal{G}_1}) = \mathcal{N}(\alpha L_{\mathcal{G}_1})$ *for all positive scalars α.*

The addition operation of two linear consensus networks is equivalent to the edge union operation on the underlying graphs of the two networks. The scalar multiplication operation of a linear consensus network is equivalent to scaling the weight function of the underlying graph. Let us denote $L_{\mathcal{G}}^{\dagger}$ to be the Moore-Penrose pseudo-inverse of $L_{\mathcal{G}}$, which is a square, symmetric, doubly-centered and positive semidefinite matrix.

Definition 2 (Convex Systemic Measures) *For a given space of linear consensus networks $\mathfrak{N}(x_0)$, a systemic measure is an operator $\rho : \mathfrak{N}(x_0) \to \mathbb{R}_+$ with the following properties:*

(i) Positive homogeneity of degree -1: $\rho\big(\kappa \mathcal{N}(L_{\mathcal{G}_1})\big) = \kappa^{-1} \rho\big(\mathcal{N}(L_{\mathcal{G}_1})\big)$ *for all $\kappa > 0$,*

(ii) Monotonicity: *If $L_{\mathcal{G}_1}^{\dagger} \preceq L_{\mathcal{G}_2}^{\dagger}$ then $\rho\big(\mathcal{N}(L_{\mathcal{G}_1})\big) \leq \rho\big(\mathcal{N}(L_{\mathcal{G}_2})\big)$,*

(iii) Convexity: $\rho\big(\mathcal{N}(\alpha L_{\mathcal{G}_1} + (1-\alpha)L_{\mathcal{G}_2})\big) \leq \alpha \rho\big(\mathcal{N}(L_{\mathcal{G}_1})\big) + (1-\alpha)\rho\big(\mathcal{N}(L_{\mathcal{G}_2})\big)$,

for all $\mathcal{N}(L_{\mathcal{G}_1}), \mathcal{N}(L_{\mathcal{G}_2}) \in \mathfrak{N}(x_0)$ and all $0 \leq \alpha \leq 1$.

The monotonicity property implies that a systemic measure ρ is *subadditive* over the set of all linear consensus networks. This property can be interpreted as a fundamental tradeoff between systemic measures and sparsity of the underlying graph. If we add more edges to an existing graph, the value of the systemic measure will decrease [3, 4]. The homogeneity property implies that among all graphs with identical interconnection topologies, the ones with larger (stronger) coupling weights have smaller systemic measures.

Zeta function based convex systemic measures: These important class of systemic measures are defined using the *spectral zeta function* of Laplacian eigenvalues of the underlying graph as follows

$$\rho_{\varsigma_p}\big(\mathcal{N}(L_{\mathcal{G}})\big) := k \left[\zeta_p(L_{\mathcal{G}})\right]^{\frac{1}{p}},$$ (4)

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for all $1 \leq p \leq \infty$ and $k > 0$, where the corresponding zeta function of the dynamical network (1)-(2) with Laplacian eigenvalues λ_i for $i = 2, \ldots, n$ is defined by

$$\zeta_p(L_{\mathcal{G}}) := \mathbf{Tr}\big((L_{\mathcal{G}}^{\dagger})^p\big) = \sum_{i=2}^{n} \lambda_i^{-p}, \tag{5}$$

where $\mathbf{Tr}(.)$ is the matrix trace operator. In our results, we show that (4) is a convex systemic measure according to Definition 2 that includes several well–known performance and robustness measures as its special cases [1, 3]. When $p \to \infty$ and $k = 1$, the systemic measure reduces to λ_2^{-1}, where λ_2 is the algebraic connectivity of graph $\mathcal{G}$. This systematic measure approximately measures the convergence time of the consensus process. For $p = 1$ and $k = \frac{1}{2}$, the systemic measure (4) is equal to the square of the $\mathcal{H}_2$–norm of the first–order linear consensus network (see equality (3)), and when $p = 2$ and $k = \frac{1}{\sqrt{2}}$, its value is equal to the $\mathcal{H}_2$–norm of a second–order linear consensus network [1].

Practical interpretations of systemic measures: In [26], we show that $\mathcal{H}_2$–based systemic measures have interesting physical interpretations. In a synchronous power grid subject to a random initial condition (disturbance) , the corresponding $\mathcal{H}_2$–based systemic measure is equal to the total resistive power loss in the network. This implies that in order to render all states to the consensus state some resistive power must be wasted throughout the network. Our results show that the least achievable values of the resistive power loss in a power grid is lower bounded by some constants that only depend on the characteristics of the underlying graph of the network. Another result in our paper [26] shows that $\mathcal{H}_2$–based systemic measure for a group of autonomous vehicles participating in a formation is equal to the total kinetic energy required to render the entire group to the consensus state. We are currently investigating to find practical interpretations of various systemic measures in dynamical networks.

Remark 1 *The value of systemic measure (4) for isospectral graphs, graphs with identical set of eigenvalues, are equal. This is one of our research thrusts for next year.*

The concept of convex systemic measure can be extended further by considering the following class of measures that are defined based on notion of Schur-convex functions.

Definition 3 (Schur–convex systemic measures) *For a given space of linear networks $\mathfrak{N}(x_0)$, a schur–convex systemic measure is an operator $\rho : \mathfrak{N}(x_0) \to \mathbb{R}_+$ that satisfies monotonicity property (ii) and convexity property (iii) in Definition 2 and is also orthogonally invariant, i.e.,*

$$\rho\big(\mathcal{N}(L_{\mathcal{G}})\big) = \rho\big(\mathcal{N}(U L_{\mathcal{G}} U^T)\big),$$

for all orthogonal matrices $UU^T = U^T U = I_n$.

$\mathcal{H}_p$–**based Schur–convex systemic measures:** For a given linear consensus network $\mathcal{N}(L_{\mathcal{G}}) \in \mathfrak{N}(x_0)$, suppose that $G(s) \in \mathbb{C}^{n \times n}$ is the transfer function of $\mathcal{N}(L_{\mathcal{G}})$ from disturbance input to its performance output. Our research results in [3] show that the following class of performance and robustness measures are Schur–convex systemic measures according to Definition 3:

$$\rho_{\mathcal{H}_p}\big(\mathcal{N}(L_{\mathcal{G}})\big) := \|G\|_{\mathcal{H}_p} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\|G(j\omega)\|_{p^*}^p \, d\omega\right)^{1/p}, \tag{6}$$

where the Schatten p-norm for every $1 \leq p \leq \infty$ is defined using the singular values of $G(j\omega)$ by $\|G(j\omega)\|_{p^*}^p = \sum_{i=1}^{n} \sigma_i\big(G(j\omega)\big)^p$. When $p = \infty$, this systemic measure is equal to λ_2^{-1}, which is in

fact the input-output $\mathcal{H}_\infty$–norm of the network. This measure can be viewed as the maximum system gain when inputs are taken over all measurable signals with finite energy. For a given linear consensus network $\mathcal{N}(L_\mathcal{G}) \in \mathfrak{N}(x_0)$, the $\mathcal{H}_p$–based systemic measure (6) characterizes performance and robustness properties of a network from an input–output system theoretic perspective. On the other hand, the zeta–function–based systemic measure (4) can be related to structural properties of the underlying graph of the network . Our research results [1,3] show that there is an inherent relationship between these two class of systemic measures and each one can be calculated by knowing the other one.

Schur–convex systemic measures generated by Schur–convex sums: The second important class of Schur–convex systemic measures is generated by sums of convex decreasing functions (also known as Schur–convex sums) of Laplacian eigenvalues [3]. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ is a decreasing convex function. For every $\mathcal{N}(L_\mathcal{G}) \in \mathfrak{N}(x_0)$, the class of measures that are defined by

$$\rho_{\Sigma f}\big(\mathcal{N}(L_\mathcal{G})\big) \;=\; \sum_{i=2}^n f(\lambda_i), \tag{7}$$

are Schur–convex systemic measure. For example, all measures generated by $f(\lambda) = \lambda^{-p}$ fall into this category. This includes $\mathcal{H}_2$–norm of the first–order and second–order linear consensus networks correspond to convex functions for $p = 1, 2$. Another important class of Schur-convex systemic measures is the class of entropy-based measures for linear consensus networks that are generated by $f(\lambda) = -\log\lambda$, which can be interpreted as a measure of uncertainty.

Our research results [1, 3] show that convex and Schur–convex systemic measures can be related to the structural properties of the underlying graph of dynamical networks. In some applications, it may not be possible to express explicitly the corresponding systemic measure in terms of the structural properties of the underlying graph of the network. Therefore, we have developed a novel methodology to obtain tight lower and upper bounds for the class of systemic measures. In the following, some of the key ideas behind our approach along with some selected results are explained.

1.1 Fundamental Limits on the Best Achievable Systemic Performance and Robustness

For a given dynamical network $\mathcal{N}(L_\mathcal{G}) \in \mathfrak{N}(x_0)$ with underlying unweighted graph $\mathcal{G}$ that consists of n nodes, the maximal and minimal values of the zeta–function–based systemic measure are bounded by

$$\frac{(n-1)^{1.5}}{2\sqrt{\delta_1 + \delta_2}} \;\leq\; \rho_{\zeta_1}\big(\mathcal{N}(L_\mathcal{G})\big) \;\leq\; \frac{n-1}{2n}\left[1 + \left(\binom{n}{2} - m\right)\mathbf{diam}(\mathcal{G})\right], \tag{8}$$

where m is the total number of edges, d_i is the degree of node i, and $\delta_\alpha := \sum_{i=1}^n d_i^\alpha$ for $\alpha = 1, 2$. The diameter of $\mathcal{G}$ is denoted by $\mathbf{diam}(\mathcal{G})$, which is basically the longest shortest path in the graph. This result was reported in [1] that shows explicitly how the value of a systemic measure depends the *microscopic* (e.g., degree nodes) and *macroscopic* (e.g., graph diameter and total number of edges) features of the underlying graph of the network. According to (3), we can quantify fundamental limits on the best achievable $\mathcal{H}_2$–norm of a dynamical network with arbitrary interconnection topology as follows

$$\sqrt{\frac{1}{2} - \frac{1}{2n}} \;\leq\; \rho_{\mathcal{H}_2}\big(\mathcal{N}(L_\mathcal{G})\big) \;\leq\; \sqrt{\frac{n^2 - 1}{12}}. \tag{9}$$

These bounds represent *hard limits* and scale with the size of the network. The lower bound is achieved iff $\mathcal{G}$ is a complete graph, and the upper bound is achieved iff $\mathcal{G}$ is a path (chain) graph. One can extract

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more detailed information about performance and robustness properties of dynamical networks by exploiting the specific structural properties of their underlying graphs. For instance, let us consider the class of linear consensus networks with at least five nodes and tree interconnection topologies. The $\mathcal{H}_2$–norm systemic measure of such networks is always worse (greater) than $(n-1)/\sqrt{2n}$ and always better (less) than $\sqrt{(n^2-1)/12}$. The lower hard limit is achieved iff $\mathcal{G}$ is a star graph, and the upper hard limit is achieved iff $\mathcal{G}$ is a path graph. These hard limits imply that the $\mathcal{H}_2$–based systemic measure of linear consensus networks with tree structures scale with $\mathcal{O}(\sqrt{n})$ as $n \to \infty$.

The inherent fundamental limits on performance and robustness in dynamical networks can also be quantified in terms of macroscopic features of the network, e.g., spectral properties of the underlying graph. We explain one of our interesting results by limiting our discussion to the class of Schur–convex systemic measures (7) with an additional assumption that $f(\lambda) \to 0$ as $\lambda \to \infty$. For a given dynamical network $\mathcal{N}(L_\mathcal{G}) \in \mathfrak{N}(x_0)$, we construct a new dynamical network by only adding at most k new arbitrary weighted edges to the underlying weighted graph. If we denote the resulting network by $\mathcal{N}(L_\mathcal{G}^{new})$, then there is a fundamental limit on the best achievable values for the systemic measure of the new dynamical network, which is quantified by

$$\rho_{\Sigma f}\big(\mathcal{N}(L_\mathcal{G}^{new})\big) \geq \sum_{i=k+2}^{n} f(\lambda_i^{old}), \tag{10}$$

where λ_i^{old} for $i = k+2, ..., n$ are the last $n-k-1$ largest Laplacian eigenvalues of the original graph of the network before adding new edges. This result implies that the performance and robustness of a dynamical network cannot be improved beyond some limits by increasing the connectivity of the underlying graph. The lower *hard limit* in (10) only depends on some of the Laplacian eigenvalues of the original dynamical network and is independent of the weights of the new edges.

The existence of hard limits in large–scale dynamical networks motivates us to discover methodologies that allow us to improve their performance and robustness by optimizing the corresponding systemic measures and trying to reach the best achievable hard limits. In [31], we build upon these notions and results to investigate a general form of combinatorial problem of growing a linear consensus network via minimizing a given systemic performance measure. Two efficient polynomial-time approximation algorithms are devised to tackle this network synthesis problem: a linearization-based method and a simple greedy algorithm based on rank-one updates. Several theoretical fundamental limits on the best achievable performance for the combinatorial problem is derived that assist us to evaluate optimality gaps of our proposed algorithms. A detailed complexity analysis confirms the effectiveness and viability of our algorithms to handle large-scale consensus networks.

1.2 Quantification of Tradeoffs Between Sparsity and Systemic Measures

The underlying dynamical structure of a network depends on coupling structure among the subsystems which are usually imposed by physical laws or global objectives. For example in linear consensus network (1)-(2), the Laplacian matrix represents coupling structure in the closed-loop system, which is the result of sparse information structure in the underlying controller array. In this research thrust, we focused on revealing foundational role of sparsity features of the underlying graph of networks in emergence of severe theoretical fundamental limits and tradeoffs on the resulting performance and robustness in large-scale dynamical networks. Our goal was to discover and quantify a variety of mathematical (Heisenberg-like) inequalities explaining such fundamental limits and tradeoffs. In our research results, we characterize the inherent tradeoff between some classes of systemic measures and sparsity features of linear consensus networks with unweighted underlying graphs. Let us denote the adjacency matrix of the underlying graph of

network (1)-(2) by $A_{\mathcal{G}}$. We define the total number of nonzero elements of the adjacency matrix, which is denote by $\|A_{\mathcal{G}}\|_0$, as our *sparsity measure*. We quantify the resulting fundamental tradeoffs in terms of this sparsity measure in multiplicative and additive forms by

$$\rho_{\zeta_1}(L_{\mathcal{G}})\,\|A_{\mathcal{G}}\|_0 \;\geq\; \frac{(n-1)^2}{4} \qquad \text{and} \qquad \left(\frac{2\rho_{\zeta_1}(L_{\mathcal{G}})-1}{\mathbf{diam}(\mathcal{G})}\right) + \|A_{\mathcal{G}}\|_0 \;\leq\; \frac{n(n-1)}{2},$$

where $\mathbf{diam}(\mathcal{G})$ is the diameter of the underlying graph and n is the size of the network. We interpret the implications of these inequalities by considering two scenarios for dynamical networks with the same size. The inequality in the left asserts that the minimum (best) achievable levels of zeta-based systemic measure for sparse networks is higher. For all dynamical networks with identical diameters, the inequality in the right implies that the maximum (worst) achievable levels of systemic measures for networks with more edges is smaller. Among all networks with identical number of edges, the ones with larger diameters have higher values of systemic measures. Previously, we showed that $\rho_{\zeta_1}(L_{\mathcal{G}}) \;=\; \rho_{\mathcal{H}_2}(L_{\mathcal{G}})^2$. Therefore, the above fundamental inequalities explain the quality of disturbance propagation in terms of sparsity features of linear consensus networks.

In [31], we extend this proposed methodology to broader classes of systemic measures. The key idea of our approach is based on the monotonicity property of systemic measures. This property implies that by improving connectivity of the underlying graph the value of systemic measure decreases. We have developed a rigorous mathematical framework to quantify the interplay between systemic measures and sparsity more explicitly using *tight* inequalities. In [20, 33], we consider general linear dynamical networks on graphs (beyond consensus networks) and investigate various roles of sparsity in emergence of fundamental limits and tradeoffs on stability and performance.

2 Sparsity and Spatial Localization Measures for Distributed Networks

The research results presented in this section are pushing the known boundaries in spatially distributed systems towards new methodologies to study sparsity and spatial localization in large-scale dynamical network in a systematic manner. The following discussion is based on our recent work reported in [27].

The omnipresence class of spatially decaying systems. Typical examples of spatially decaying systems include linearized models of spatially distributed power networks with sparse interconnection topologies, linearized models spatially distributed networks of autonomous agents in formation, spatially discretized models of infinite-dimensional systems with partial differential operators with constant coefficients, and many more. In our research results, we assume that the underlying spatial domain is discrete, i.e., $\mathbb{G} = \mathbb{Z}$ and the state-space operators of system

$$\dot{\psi}(t) \;=\; A\psi(t) + Bu(t), \tag{11}$$
$$y(t) \;=\; C\psi(t) + Du(t) \tag{12}$$

are infinite-dimensional matrices. In order to understand the asymptotic behavior of localization techniques, we quantify spatial decay property by means of *coupling weight functions*. The Euclidean norm is denoted by $|.|$. A positive nondecreasing function w on $\mathbb{Z}$ is a weight function if it satisfies: $w(k) \geq 1$ and $w(k) = w(-k)$ for all $k \in \mathbb{Z}$.

Definition 4 *For $0 < q \leq 1$, the coupling weight function w is called admissible if there exist a companion weight function u, an exponent $\theta \in (0,1)$, and a positive constant D such that*

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(a) $w(k + l) \le w(k)\, u(l) + u(k)\, w(l)$ *for all* $k, l \in \mathbb{Z}$,

(b) $\inf_{\tau > 0} \left\| u \chi_{[-\tau, \tau]} \right\|_{\frac{2}{2-q}} + t \left\| wu^{-1}(1 - \chi_{[-\tau, \tau]}) \right\|_{\infty} \le D\, t^{1-\theta}$ *for all* $t \ge 1$.

The function $\chi_{[-\tau, \tau]}$ is the characteristic function on the interval $[-\tau, \tau]$. The first condition implies that w is submultiplicative, i.e., $w(k + l) \le w(k)\, w(l)$. Typical examples of admissible coupling weight functions that appear in various applications are subexponential weights $w(k) = e^{\alpha |k|^{\beta}}$ for $\alpha > 0$ and $0 \le \beta < 1$, and polynomial weights $w(k) = (1 + |k|)^{s}$ for $s \ge 0$. More general weights are mixtures of the form $w(k) = e^{\alpha |k|^{\beta}} (1 + |k|)^{s} \left(\log(e + |k|) \right)^{\sigma}$ for $\sigma \ge 0$.

Definition 5 *For a fixed exponent $0 < q \le \infty$ and admissible coupling weight function w, the class of matrices $\mathcal{S}_{q,w}(\mathbb{Z})$ consists of all spatially decaying matrices $A = (a_{ij})_{i,j \in \mathbb{Z}}$ such that A is bounded with respect to the matrix norm*

$$\|A\|_{\mathcal{S}_{q,w}(\mathbb{Z})} := \max \left\{ \sup_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |a_{ij}|^{q} w(i - j)^{q} \right)^{1/q}, \sup_{j \in \mathbb{Z}} \left(\sum_{i \in \mathbb{Z}} |a_{ij}|^{q} w(i - j)^{q} \right)^{1/q} \right\}, \tag{13}$$

for $0 < q < \infty$, and by

$$\|A\|_{\mathcal{S}_{\infty,w}(\mathbb{Z})} := \sup_{i,j \in \mathbb{Z}} |a_{ij}| w(i - j). \tag{14}$$

for $q = \infty$. In fact, the matrix norm (13) is a quasi-norm for $0 < q < 1$.

The matrix norm inherits the key submultiplicative property from the coupling weight function, i.e., $\|AB\|_{\mathcal{S}_{q,w}(\mathbb{Z})} \le \|A\|_{\mathcal{S}_{q,w}(\mathbb{Z})} \|B\|_{\mathcal{S}_{q,w}(\mathbb{Z})}$. This property turns $\mathcal{S}_{q,w}(\mathbb{Z})$ into an interesting mathematical object so called q-Banach algebra for $0 < q \le 1$ and Banach algebra for $1 \le q \le \infty$ that is closed under addition and multiplication operations for all $0 < q \le \infty$, i.e., if A, B belong to $\mathcal{S}_{q,w}(\mathbb{Z})$, then $A + B \in \mathcal{S}_{q,w}(\mathbb{Z})$ and $AB \in \mathcal{S}_{q,w}(\mathbb{Z})$. One of the fundamental properties of the class of spatially decaying matrices $\mathcal{S}_{q,w}(\mathbb{Z})$ is that it is *inverse-closed* in $\mathcal{B}(\ell^{2}(\mathbb{Z}))$ (the Hilbert space of all bounded matrices on $\ell^{2}(\mathbb{Z})$), i.e., if for every $A \in \mathcal{S}_{q,w}(\mathbb{Z})$ that is invertible on $\ell^{2}(\mathbb{Z})$), we have that $A^{-1} \in \mathcal{S}_{q,w}(\mathbb{Z})$ [27].

Definition 6 *For a fixed $0 < q \le \infty$, a linear system (11)-(12) is called spatially decaying on $\mathbb{Z}$ with respect to an admissible coupling weight function w if its state-space matrices A, B, C, D belong to $\mathcal{S}_{q,w}(\mathbb{Z})$.*

The class of spatially decaying linear systems is closed under basic system operations such as series, parallel, and feedback interconnections.

Inherent sparse information structure of linear-quadratic regulators (LQR). In our first major result, we consider the class of spatially decaying linear systems defined over $\mathcal{S}_{q,w}(\mathbb{Z})$ for $0 < q \le \infty$ (Definition 6). One of the fundamental problems in the context of spatially distributed systems is to investigate structural properties of the linear-quadratic regulator (LQR) problem for spatially decaying system (11)-(12) with cost functional

$$J(\psi_{0}, u) = \int_{0}^{\infty} \left(\psi^{*} Q \psi + u^{*} R u \right) dt. \tag{15}$$

Suppose that weight matrices Q and R are self-adjoint and positive definite and (A, B) is exponentially stabilizable and (Q, A) is exponentially detectable in $\mathcal{B}(\ell^{2}(\mathbb{Z}))$ (the Hilbert space of all bounded matrices on $\ell^{2}(\mathbb{Z})$). The LQR feedback control law is given by $u = -K\psi = -R^{-1}B^{*}X\psi$ where X is the unique positive-definite solution of the algebraic Riccati equation

$$A^{*}X + XA - XBR^{-1}B^{*}X + Q = 0. \tag{16}$$

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The spatial decay properties of the quadratically optimal controllers and filtering problems for spatially decaying systems are closely related to the algebraic properties of the corresponding algebraic Riccati equations. We have the following fundamental result for the class of spatially decaying systems with bounded operators.

Theorem 1 ([27]) *For a fixed exponent $0 < q \leq \infty$, suppose that A, B, Q, R belong to $\mathcal{S}_{q,w}(\mathbb{Z})$. Then the unique positive-definite solution of the algebraic Riccati equation (16) also belongs to $\mathcal{S}_{q,w}(\mathbb{Z})$. Furthermore, the closed-loop matrix $A - R^{-1}B^*X$ generates exponentially stable semigroup on $\mathcal{S}_{q,w}(\mathbb{Z})$.*

The outcome of Theorem 1 states that the LQR feedback control law $K = \left(K_{ij}\right)_{i,j \in \mathbb{Z}}$ is **spatially localized**, i.e.,

$$|K_{ij}| \leq C_0 \, w(i-j)^{-1} \tag{17}$$

where $C_0 = \|K\|_{\mathcal{S}_{q,w}(\mathbb{Z})}$ is a finite number. This implies that the underlying information structure of the optimal control law is **sparse** in space and each local controller needs to receive information only from some neighboring subsystems (and not from the entire spatially distributed system). The implications of our results for spatially decaying systems lay foundations of a general theory to discover fundamental limits of feedback control strategies in spatially distributed systems due to sparse information structures.

Fundamental limits on best achievable degrees of spatial localization. In order to characterize a fundamental limits on interplay between stability margins and optimal performance loss in spatially distributed systems, we approximate the corresponding LQR feedback gain K by a sparse feedback gain $K^{\mathfrak{T}-\text{nearest}}$ for a given truncation length $\mathfrak{T} > 0$ as follows

$$\left(K^{\mathfrak{T}-\text{nearest}}\right)_{ij} = \begin{cases} K_{ij} & \text{if } |i-j| \leq \mathfrak{T} \\ 0 & \text{if } |i-j| > \mathfrak{T} \end{cases}. \tag{18}$$

By applying small-gain theorem, we can characterize a fundamental limit in the form of a lower bound for *stabilizing truncation lengths* for different admissible coupling weight functions [27]. For the subexponential coupling weight function with parameters $\alpha > 0$ and $0 \leq \beta < 1$, the sparse feedback controller $K^{\mathfrak{T}-\text{nearest}}$ is exponentially stabilizing if the truncation length $\mathfrak{T}$ satisfies the following inequality

$$\mathfrak{T} > \mathfrak{T}_s = \left(\frac{1}{\alpha} \log \left(\|K\|_{\mathcal{S}_{q,w}(\mathbb{Z})} \|B\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \sup_{\text{Re}(s)>0} \sigma_{\max}(G(s)) \right) \right)^{\frac{1}{\beta}}, \tag{19}$$

where $G(s) = (sI - (A + BK))^{-1}$ is the transfer function of the closed-loop system. A fundamental *tradeoff* emerges between truncation length and performance loss. We define the performance loss as the expected value of the quantity $|J(x_0, Kx) - J(x_0, K^{\mathfrak{T}-\text{nearest}}x))|$ over all random Gaussian initial conditions with zero mean and identity covariance matrix. Our results show that the performance loss asymptotically converges to zero as $\mathfrak{T} \to \infty$ [27].

Fundamental tradeoffs between sparsity and spatial locality. Let us consider an ideal sparsity measure whose value returns the maximum number of nonzero entries in all rows and columns of a matrix, i.e.,

$$\|K\|_{\mathcal{S}_{0,1}(\mathbb{Z})} := \max \left\{ \sup_{i \in \mathbb{Z}} \|K(i,.)\|_{\ell^0(\mathbb{Z})}, \ \sup_{j \in \mathbb{Z}} \|K(.,j)\|_{\ell^0(\mathbb{Z})} \right\},$$

where $K(i,.)$ is the i'th row and $K(.,j)$ the j'th column of matrix K. The $\ell^0(\mathbb{Z})$ quasi-norm of a vector is the cardinality of the set of its nonzero entries. The $\mathcal{S}_{0,1}$-measure (i.e., the value of $\|.\|_{\mathcal{S}_{0,1}}$) is specifically
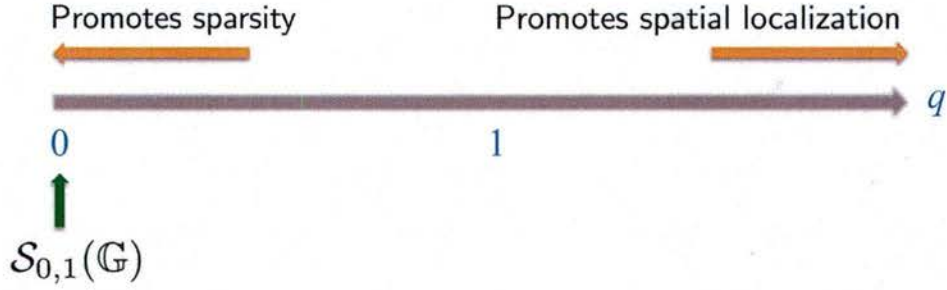
Figure 2: For $1 \leq q \leq \infty$, $\mathcal{S}_{q,w}(\mathbb{Z})$ is a Banach algebra. However, $\mathcal{S}_{q,w}(\mathbb{Z})$ for $0 < q < 1$ is a quasi–Banach algebra. The analysis of spatially distributed systems which are defined over q-Banach algebras for $0 < q \leq 1$ are much more subtle than those systems that are defined over Banach algebras.

tailored for spatially distributed systems as it is a suitable measure for sparsity and spatial localization simultaneously. The $\mathcal{S}_{q,w}$–measure (i.e., the value of $\|.\|_{\mathcal{S}_{q,w}(\mathbb{Z})}$) of a matrix can be considered as an asymptotic approximation of its $\mathcal{S}_{0,1}$–measure, since the following limit holds

$$\lim_{q \to 0} \|K\|_{\mathcal{S}_{q,w}(\mathbb{Z})}^{q} = \|K\|_{\mathcal{S}_{0,1}(\mathbb{Z})}, \tag{20}$$

for all spatially decaying matrices with bounded entries. This implies that the class of spatially decaying matrices $\mathcal{S}_{q,w}(\mathbb{Z})$ for $0 < q \leq 1$ are sparse and enjoy certain interesting diagonal decay properties.

An important fundamental tradeoff emerges when exponent q varies from zero to infinity. On one hand, when q tends to large numbers and gets closer to ∞, the $\mathcal{S}_{q,w}$–measure asymptotically approximates $\mathcal{S}_{\infty,w}$–measure and becomes an ideal spatial locality measure. On the other hand, when q tends to small numbers and gets closer to zero, the $\mathcal{S}_{q,w}$–measure asymptotically approximates $\mathcal{S}_{0,1}$–measure and becomes an ideal sparsity measure. We prove that there exists an exponent $q^* < 1$ such that the $\mathcal{S}_{q,w}(\mathbb{Z})$–measure approximates the ideal sparsity measure $\mathcal{S}_{0,1}$ in probability for all $0 < q < q^*$. In order to present an explicit and sensible result, we focus on the class of exponentially decaying random matrices of the form

$$\mathcal{R}_{\sigma,\delta}(\mathbb{Z}) = \left\{ K = \left(r_{ij}\, e^{-\left(\frac{|i-j|}{\sigma}\right)^{\delta}} \right)_{i,j \in \mathbb{Z}} \ \middle| \ r_{ij} \sim \mathbf{U}(-1,1) \right\}$$

for some given parameters $\sigma > 0$ and $\delta \in (0,1)$. The coefficients r_{ij} are drawn from the continuous uniform distribution $\mathbf{U}(-1,1)$. For a given truncation threshold $0 < \epsilon < 1$, the sparsity indicator function for a random subexponentially decaying matrix $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ is defined by

$$\Psi_{K,w}(q,\epsilon) := \frac{\|K\|_{\mathcal{S}_{q,w}}^{q}}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1}, \tag{21}$$

where $\lfloor . \rfloor$ is the floor function. For proper values of exponent q and parameters in the coupling weight function w, the sparsity indicator function can take values sufficiently close to one. This implies that matrix $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ can be approximated by a sparse matrix whose $\mathcal{S}_{0,1}$–measure is close to $\|K\|_{\mathcal{S}_{q,w}}^{q}$ in probability.

Theorem 2 ([27]) *Consider the class of matrices $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ and the subexponential coupling weight function w' with parameters $\sigma' = \frac{\sigma}{\eta}$ for some $\delta \in (0,1)$ for some $\eta \in (0,1)$. Let us define parameter $\beta = q \ln \epsilon^{-1}$.*

Then, for every $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ we have

$$\lim_{\epsilon \to 0^+} \mathbb{P}\left\{ \left| \Psi_{K,w'}(q,\epsilon) - \gamma' \right| < \epsilon_0 \right\} = 1, \tag{22}$$

for every $\epsilon_0 > 0$, where $\gamma' := \int_0^\infty e^{-(1-\eta^\delta)t^\beta} dt$.

The result of this theorem provides an algorithm to approximate the LQR feedback gain K with a near-optimal sparse matrix. Since σ and δ are known, we can pick the positive parameters η and β such that $\gamma' \approx 1$, for example, $\gamma' = 0.95$. If we fix parameter β, the sparsity indicator function becomes a single variable function of exponent q and its value can be calculated for range of parameters $0 < q \leq 1$. Then, one can calculate an exponent q^* such that the value of the sparsity indicator function stays arbitrarily close to γ' for all $0 < q < q^*$. The number $\|K\|^{q^*}_{\mathcal{S}_{q^*,w'}}$ represents an *optimal tradeoff* between sparsity and spatial localization measures. Furthermore, we can extract more information from exponent q^* and compute an optimal truncation length using the formula $\mathfrak{T}_0 = \sigma \sqrt[\delta]{\frac{\beta}{q^*}}$. By combining this result with (19), a near-optimal stabilizing truncation length for the LQR feedback gain is given by

$$\mathfrak{T}_{\mathrm{op}} = \max\left\{ \mathfrak{T}_0, \mathfrak{T}_s \right\}. \tag{23}$$

Mathematical foundation of notions of sparsity in spatially decaying systems. The notion of sparsity in spatially decaying systems can be generalized to more abstract ground via quasi-Banach algebras, which is a complex vector space of matrices $\mathcal{A}$ on $\mathbb{Z}$ equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ for $0 < q \leq 1$. The quasi-Banach algebras enjoy similar basic properties to $\mathcal{S}_{q,w}(\mathbb{Z})$, e.g., it is closed under addition and multiplication operations.

Definition 7 ([27]) *For $0 < q \leq 1$, a q-Banach algebra $\mathcal{A}$ of matrices equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ is called proper if it is: (i) closed under the complex conjugate operation, (ii) a subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}))$ and continuously embedded with respect to it, and (iii) a differential Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}))$.*

Our results show that a proper q-Banach algebra is inverse-closed. This enable us to define the class of linear systems over proper q-Banach algebra and study their exponential stability. Moreover, we prove that, under standard assumptions, if the coefficients of the algebraic Riccati equation belong to a proper q-Banach algebra $\mathcal{A}$, then $X \in \mathcal{A}$.

2.1 Interplay Between Spatial Localization Measures and Systemic Measures

In our research results [26], we show that for the class of linear consensus networks (1)-(2) with $n \geq 3$, the following fundamental limit emerges between the ideal sparsity measure and zeta function based systemic measures

$$\rho_{\zeta_1}(L_{\mathcal{G}}) \|A_{\mathcal{G}}\|_{\mathcal{S}_{0,1}} \geq \frac{n-1}{4}, \tag{24}$$

where $A_{\mathcal{G}}$ is the adjacency matrix of the underlying unweighted graph of the network. For networks with identical number of nodes, this fundamental tradeoff asserts that by improving local connectivity in a network the minimum (best) achievable level of the systemic measure decreases.

The PI believes that theses results can be extended to quantify fundamental tradeoffs between systemic measures and the class of $\mathcal{S}_{q,w}$–measures for sparsity and spatial localization. This is extremely important as it will reveal several key network design principles and show the effects of spatial localization on global performance and robustness of spatially distributed dynamical networks. This objective is particularly challenging as the class of $\mathcal{S}_{q,w}$–measures for $0 < q < 1$ are nonconvex matrix norms and require us to develop our framework based on tools from quasi-Banach algebras.

3 Network Sparsification with Guaranteed Systemic Performance Measures

A sparse consensus network is one whose number of coupling links is proportional to its number of sub-systems. Optimal design problems for sparse consensus networks are more amenable to efficient optimization algorithms. More importantly, maintaining such networks are usually more cost effective due to their reduced communication requirements. Therefore, approximating a given dense consensus network by a suitable sparse network is an important analysis and synthesis problem. In this chapter, we develop a framework to produce a sparse approximation of a given large-scale network with guaranteed performance bounds using a nearly-linear time algorithm. First, the existence of a sparse approximation of a given network is proven. Then, we present an efficient and fast algorithm for finding a near-optimal sparse approximation of a given network. Finally, several examples are provided to support our theoretical developments.

In [8, 35], we specifically addressed the following network design problem: given a linear consensus network with an undirected connected underlying graph, the *network sparsification* problem seeks to re-place the coupling graph of the original network with a reasonably sparser subgraph so that the behavior of the original and the sparsified networks is similar in an appropriately defined sense. Such situations arise frequently when real-world large-scale dynamical networks need to be simulated, controlled or redesigned using efficient computational tools that are specifically tailored for optimization problems with sparse struc-tures. We develop a general methodology that computes sparsifiers of a given consensus network using a nearly-linear time $\tilde{\mathcal{O}}(m)$[1] algorithm with guaranteed systemic performance bounds, where m is the number of links. Unlike other existing work on this topic in the literature, our proposed framework: (i) works for a broad class of systemic performance measures including $\mathcal{H}_2$-based performance measures, (ii) does not involve any sort of relaxations such as ℓ_0 to ℓ_1, (iii) provides guarantees for the existence of a sparse solu-tion, (iv) can partially sparsify predetermined portions of a given network; and most importantly, (v) gives guaranteed systemic performance certificates.

While our approach is relied on several existing works in algebraic graph theory, our control theoretic contributions are threefold. First, we show that every given linear consensus network has a sparsifier net-work such that the two networks yield comparable performances with respect to any systemic performance measure. Second, we develop a framework to find a sparse approximation of large-scale consensus net-works using a fast randomized algorithm. We note that while the coupling graph of the sparsified network is a subset of the coupling graph of the original network, the weights of links (the strength of each coupling) in the sparsified network are adjusted accordingly to reach predetermined levels of systemic performance. Third, we prove that our development can also be applied for partial sparsification of large-scale networks, which means that we can sparsify a prespecified subgraph of the original network to find an approximation of the network with fewer links. This is practically plausible as our algorithm can be spatially localized, if necessary, and it does not require to receive information of the entire coupling graph of the network.

We develop a sparsification framework for the class of linear consensus networks governed by (1)-(2) that provides performance guarantees with respect to a general class of systemic measures. First, we introduce a notion of approximation for the class of consensus networks.

Definition 8 *For a fixed constant $\epsilon \in [0, 1]$, a consensus network $\mathcal{N}(L_s)$ is ϵ-approximation of $\mathcal{N}(L)$ if and only if*

$$(1 - \epsilon)^\alpha \leq \frac{\rho(L)}{\rho(L_s)} \leq (1 + \epsilon)^\alpha, \tag{25}$$

for every homogenous convex systemic measure $\rho : \mathfrak{L}_n \to \mathbb{R}_+$ of order $\alpha > 0$.

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[1]We use $\tilde{\mathcal{O}}(.)$ to hide poly log log terms from the asymptotic bounds. Thus, $f(n) \in \tilde{\mathcal{O}}(g(n))$ means that there exists $k > 0$ such that $f(n) \in \mathcal{O}(g(n) \log^k g(n))$.

Lemma 1 $\mathcal{N}(L_s)$ is an ϵ-approximation of $\mathcal{N}(L)$ if and only if the following inequalities hold

$$(1 - \epsilon)L \preceq L_s \preceq (1 + \epsilon)L. \tag{26}$$

This result is fundamental as it enables us to take advantage of monotonicity property of systemic measures in our approximations.

Existence of ϵ-Approximations: Among all ϵ-approximation of a given consensus network with a dense coupling graph, we are interested in its sparsifiers, i.e., those networks with sparse coupling graphs.

Definition 9 $\mathcal{N}(L_s)$ is a (ϵ, d)-sparsifier of a given network $\mathcal{N}(L)$ with n nodes if the following conditions hold:
1. $\mathcal{N}(L_s)$ is an ϵ-approximation of network $\mathcal{N}(L)$; and
2. the coupling graph of $\mathcal{N}(L_s)$ has at most $dn/2$ links (i.e., feedback gains).

The second condition implies that the average number of connected links to nodes in the sparsifier is at most d. In the following theorem, we show existence of sparsifiers for every given consensus network.

Theorem 3 Suppose that a consensus network $\mathcal{N}(L)$ with coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ and $d > 1$ are given. Then, there exists a consensus network $\mathcal{N}(L_s)$ with coupling graph $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s, w_s)$ such that $\mathcal{N}(L_s)$ is a $(\frac{2\sqrt{d}}{d+1}, 2d)$-sparsifier of $\mathcal{N}(L)$ and $\mathcal{E}_s \subset \mathcal{E}$.

Specifically, Theorem 3 shows that every given linear consensus network with dynamics (1)-(2) has a sparse consensus network such that the two networks yield comparable performances with respect to any systemic performance measure $\rho : \mathfrak{L}_n \to \mathbb{R}_+$.

In our next result, we show that every consensus network has a sparse consensus network such that: (i) it yields a better systemic performance than the original network, and (ii) the total weight sum of the coupling graph of the sparsifier is controlled, i.e., it is less than a constant multiplier of that of the original network. For a given coupling graph with Laplacian matrix L and the set of links $\mathcal{E}$, we denote the total weight sum of the coupling graph by

$$w_{\text{total}}(L) := \sum_{e \in \mathcal{E}} w(e). \tag{27}$$

Theorem 4 For a given consensus network $\mathcal{N}(L)$ with coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ and every $d > 1$, there exists a consensus network $\mathcal{N}(L_s)$ with coupling graph $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s, w_s)$ that has at most dn links and $\mathcal{E}_s \subset \mathcal{E}$. Moreover, we have

$$w_{total}(L_s) \leq \left(\frac{\sqrt{d}+1}{\sqrt{d}-1} \right)^2 w_{total}(L) \tag{28}$$

and

$$\rho(L_s) \leq \rho(L),$$

for every convex systemic measure $\rho : \mathfrak{L}_n \to \mathbb{R}_+$.

We note that linear consensus networks with smaller systemic performance measure are more desirable and they exhibit better network-wide performance.

Computing Sparsifiers via Random Sampling: We employ a randomized algorithm to compute a (ϵ, d)-sparsifier of a given network. A randomized algorithm employs a degree of randomness as part of its

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logic. Randomization allows us to design provably accurate algorithms for problems that are massive and computationally expensive or NP-hard. According to our discussions in [8,35], we sample low-connectivity coupling links with high probability and high-connectivity coupling links with low probability. For a given consensus network $\mathcal{N}(L)$ with n nodes, we sample links of the coupling graph of this network M times in order to produce a $(\epsilon, 2M/n)$-sparsifier of it. Let us denote probability of selecting a link $e \in \mathcal{E}$ by $\pi(e)$ that is proportional to $w(e)r(e)$, where $w(e)$ and $r(e)$ are the weight and the effective resistance of link e, respectively. In each step of sampling, we add the selected link e to the sparsifier network with weight $w(e)/(M\pi(e))$. All details of this algorithm is explained below:

Table 1: Network Sparsification Algorithm

Algorithm:
Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$
1:    set $\mathcal{G}_s$ to be the empty graph on $\mathcal{V}$
2:    for $i = 1$ to M
3:       sample link $e \in \mathcal{E}$ with probability $\pi(e)$
add it to $\mathcal{G}_s$ with link weight $w(e)/(M\pi(e))$
4:    end for

The following theorem provides us with a proof certificate that the above randomized algorithm is capable of generating a sparse approximation of a given consensus network.

Theorem 5 *For a given consensus network $\mathcal{N}(L)$, a fixed constant $\epsilon \in (1/\sqrt{n}, 1]$ and an integer number $M = \mathcal{O}(n \log n/\epsilon^2)$, Network Sparsification Algorithm produces a $(\epsilon, 2M/n)$-sparsifier of network $\mathcal{N}(L)$ with high probability*[2].

Network Sparsification Algorithm produces a sparsifier with with $\mathcal{O}(n \log n/\epsilon^2)$ links (i.e., feedback gains) in expectation and runs in approximately linear time $\tilde{\mathcal{O}}(m)$, where m is the number of links [8,35]. To do so, good approximations of all effective resistances are needed, and Spielman and Srivastava prove that $\mathcal{O}(\log n)$ calls to a solver for symmetric diagonally dominant (SDD) linear systems can provide sufficiently good approximations to all effective resistances.

4   Performance and Robustness Measure for Time-Delay Networks

In [11], we investigate performance of noisy time-delayed linear consensus networks from a graph topological point of view. Performance of the network is measured by the square of the $\mathcal{H}_2$-norm of the system. The focus of this work is on noisy consensus networks with homogeneous time delays affecting both the agent and all its neighbors. We derive an exact expression for the performance measure of the network in terms of time delay parameter and Laplacian eigenvalues of the underlying graph of the network. It is shown that the performance measure is a convex and Schur-convex function of Laplacian eigenvalues. We characterize the network topology with optimal performance. Furthermore, we quantify a fundamental limit on the best achievable performance based on performance of the optimal topology.

In [17], we investigate topology design for optimal performance in time-delay noisy networks. Performance of the network is measured by the square of the system's $\mathcal{H}_2$-norm. Our focus was on adding new

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[2]An event holds with high probability if it holds with probability $1 - \mathcal{O}(n^{-a})$ for some $a > 0$ independent of n.

interconnections to enhance performance of the time-delay first-order connected consensus network. We discuss complexity of topology optimization for delayed networks and develop two practical methods to tackle the combinatorial eigenvalue problem without exhaustive search or eigen-decomposition. Furthermore, we compare these methods and discuss their degrees of optimality. In [32], we provide a comprehensive analysis on the effect of time-delay that reincarnates itself in the form of non-monotonicity, which results in nonintuitive behaviors of the performance as a function of graph topology. We present three methods to improve the performance by growing, re-weighting, or sparsifying the underlying graph of the network. It is shown that our proposed algorithms provide near-optimal solutions with lower complexity with respect to existing methods in literature.

In [15], we extend our focus and consider time-delay nonlinear consensus networks. We investigate stability and convergence properties of a class of nonlinear delayed consensus networks. Using tools and techniques from functional differential equations, sufficient stability conditions with respect to a common state as well as estimates on the convergence rate are derived. We characterize the limit (consensus) state for time-invariant sub-classes of these networks. More importantly, we specify under what conditions a delayed network exhibits periodic synchronized solutions. We provide sufficient conditions for existence, uniqueness and stability of this interesting phenomenon. There are several important open problems remaining from this effort.

5 Design of Sparse Feedback Controllers for Spatially Distributed Systems

In the following efforts, we implemented our theoretical findings to compute sparse feedback controllers using existing optimization algorithms.

5.1 Closed-Loop Feedback Sparsification Under Parametric Uncertainties

In [19], we consider the problem of output feedback controller sparsification for systems with parametric uncertainties. We develop an optimization scheme that minimizes the performance deterioration from that of a well-performing pre-designed centralized controller, while enhancing sparsity pattern of the feedback gain. In order to improve temporal proximity of the pre-designed control system and its sparsified counterpart, we also incorporate an additional constraint into the problem formulation such that the output of the controlled system is enforced to stay in the vicinity of the output of the pre-designed system. It is shown that the resulting non-convex optimization problem can be equivalently reformulated into a rank-constrained problem. We then formulate a bi-linear minimization problem to obtain a sub-optimal solution which satisfies the rank constraint with arbitrary tolerance. Finally, a sub-optimal sparse controller synthesis for IEEE 39-bus New England power network is used to showcase the effectiveness of our proposed method.

5.2 Output Feedback Controller Sparsification via $\mathcal{H}_2$-Approximation

In [9], we investigate the problem of optimal sparse output feedback control design for continuous linear time invariant systems. Unlike previously proposed methods where linear quadratic regulator control is modified to promote the sparsity level of the controller gain matrix, this research adopts the concept of $\mathcal{H}_2/\mathcal{H}_\infty$ control to develop an optimization program capable of synthesizing a structured sparse static controller gain for which the overall closed loop system exhibits empirical frequency characteristics resembling that of the system controlled with a pre-designed centralized controller. We, moreover, modified our optimization problem so that the control signal generated by the sparse controller falls into the vicinity of the centralized control input, in the sense of $\mathcal{L}_2$–norm. Furthermore, we show that our optimization problem can

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be equivalently reformulated into a rank constrained problem for which we propose to use a version of Alternating Direction Method of Multipliers (ADMM) as a computationally efficient algorithm to sub-optimally solve it. There are several renaming issues to address regarding complexity of existing optimization tools.

6 Systemic Performance Measures for Nonlinear Dynamical Networks

6.1 Koopman Performance Analysis of a Class of Nonlinear Dynamical Networks

In this research effort, we aimed at extending some of our systemic measures results to nonlinear dynamical networks. In [21], we have been able to utilize Koopman operator machinery to calculate a performance measures of a class of networks with nonlinear nodal dynamics. Let us consider the following Cucker-Smale type nonlinear network

$$\dot{x} = -L(x)x. \tag{29}$$

The elements of the state-dependent Laplacian matrix $L(x) = [l_{ij}(x)]$ are a nonlinear function of distance between the agents, i.e.,

$$l_{ij}(x) = f(\|x_i - x_j\|).$$

The output or observable of the network is

$$y(t) = x(t) - \mathbf{1}\bar{x}(t),$$

where $\mathbf{1}$ is the vector of all ones and $\bar{x}(t)$ is the average of all states at time $t \geq 0$. Suppose that dynamical network (29) is in its equilibrium state, i.e., all agents have agreed upon a consensus state $x^* = \alpha\mathbf{1}$ for some constant α. If we twitch the network by abruptly changing the state of the system to a random state (i.e., network (29) with a random initial condition), the required mean output energy to steer the network to its equilibrium (i.e., nominal form) is given by quantity

$$\rho_{ss}(L) = \mathbb{E}\left\{ \int_0^\infty y^T(\tau)y(\tau)d\tau \right\}. \tag{30}$$

We have developed a methodology for quantifying this performance measure in terms of the Koopman eigenfunctions of the nonlinear network, and the eigenvalues of the corresponding linearized system at the equilibrium of the network, where the eigenvalues of the linearized system are indeed Laplacian eigenvalues of the underlying graph with opposite signs. More precisely, we have

$$\rho_{ss}(L) = - \sum_{i,j=2}^{\infty} \frac{c_{ij}}{\lambda_i + \lambda_j} \phi_{ij} \tag{31}$$

where $\{\lambda_i\}_{i=1}^\infty$ is the sequence of Koopman eigenvalues, $\{\phi_i\}_{i=1}^\infty$ is the sequence of Koopman eigenfunctions, and $\{C_i\}_{i=1}^\infty$ is the sequence of Koopman modes. We show that $\lambda_1 = 0$ and $\lambda_i < 0$ for all $i > 1$ for connected coupling graphs. Our results reveal that the performance measure $\rho_{ss}(L)$ depends on the interconnection topology of the underlying graph. The reason is that Koopman eigenvalues can be represented in terms of Laplacian eigenvalues of the underlying graph. The effectiveness of our results are shown on several benchmark examples, including a Cucker-Smale type consensus network and first-order network of identical Kuramoto oscillators. Furthermore, as it is shown in Figure 4 of our paper [21], nonlinear networks with more dense coupling topology tend to have better performance features. We have gained some valuable insights from these results and identified potential difficulties of dealing with nonlinear networks. We believe that our operator theoretic approach along with advanced graph theory and some scalable approximation schemes will enable us to solve performance and robustness analysis of nonlinear networks.

6.2 Fundamental Limits on Performance of Autocatalytic Pathways

I briefly outline the basics of our method to analyze autocatalytic dynamical networks [?, ?, 34]. We will consider three scenarios. First, a minimal model of an autocatalytic system is analyzed. Then, a more general network topology with multiple intermediate metabolite reactions is considered. Third, we propose a general model for autocatalytic networks using graph theory tools. Finally, we propose a novel approach to characterize hard limits in interconnected networks of dynamical systems using duality concepts. The first step is development of a biologically motivated minimal model of autocatalytic dynamical networks that exhibits fundamental tradeoffs caused by autocatalysis. Analysis of robustness and efficiency tradeoffs for such canonical model provides a deep understanding of structural properties of autocatalytic networks as well as illustrates theoretical underpinning principles to design efficient and robust networks of dynamical systems.

Minimal model of glycolysis. The central role of glycolysis is to consume glucose and produce adenosine triphosphate (ATP), the cells energy currency. Similar to many other engineered systems whose machinery runs on its own energy product, the glycolysis reaction is autocatalytic. The ATP molecule contains three phosphate groups and energy is stored in the bonds between these phosphate groups. Two molecules of ATP are consumed in the early steps (hexokinase, phosphofructokinase/PFK) and four ATPs are generated as pyruvate is produced. PFK is also regulated such that it is activated when the adenosine monophosphate (AMP)/ATP ratio is low; hence it is inhibited by high cellular ATP concentration. This pattern of product inhibition is common in metabolic pathways.

Early experimental observations in Saccharomyces cerevisiae suggest that there are two synchronized pools of oscillating metabolites. Metabolites upstream and downstream of phosphofructokinase (PFK) have 180 degrees phase difference, suggesting that a two-dimensional model incorporating PFK dynamics might capture some aspects of system dynamics, and indeed, such simplified models qualitatively reproduce the experimental behavior. We consider a minimal system with three reactions with a single intermediate metabolite reaction (32)-(34), for which we can identify specific mechanisms both necessary and sufficient for oscillations,

$$s + \alpha y \quad \xrightarrow{f} \quad x_1 \tag{32}$$

$$x_1 \quad \xrightarrow{k_x} \quad (\alpha + \beta)y + x_1' \tag{33}$$

$$y \quad \xrightarrow{k_y} \quad \varnothing \tag{34}$$

In the first reaction, s is some precursor and source of energy for the pathway with no dynamics associated, y denotes the product of the pathway (ATP), x_1 is intermediate metabolites, x_1' is one of the by-products of the second biochemical reaction, $\varnothing$ is a null state, α is the number of y molecules that are invested in the pathway, and $\alpha + \beta$ is the number of y molecules produced. $A \xrightarrow{k} B$ denotes a chemical reaction that converts the chemical species A to the chemical species B at rate k. We choose $f(y) = \frac{Vy^q}{1+\gamma y^h}$, which is consistent with biological intuition and experimental data in the case of the glycolysis pathway, where $V > 0$ depends on s, parameter $q > 0$ captures the strength of autocatalysis, and $\gamma, h > 0$ capture the strength of inhibition. The function f is not monotone and captures the interplay between the autocatalysis and inhibition. A set of ordinary differential equations that govern the changes in concentrations x_1 and y can be written as

$$\dot{x}_1 = -k_x x_1 + \frac{Vy^q}{1+\gamma y^h} \tag{35}$$

$$\dot{y} = -k_y y + (\alpha + \beta)k_x x_1 - \frac{\alpha Vy^q}{1+\gamma y^h}, \tag{36}$$

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for $x_1 \geq 0$ and $y \geq 0$. To highlight fundamental tradeoffs due to autocatalytic structure of the system, we normalize the concentration such that steady states are $\bar{y} = 1$ and $\bar{x} = \frac{k_y}{\beta k_x}$.

Depending on values of parameters q and h, the system can have another equilibrium point which is unstable when $(\bar{x}, \bar{y})$ is stable. Our model (35)-(36) is a general version of model proposed by Doyle et. al.. In glycolysis model (35)-(36), expression $\frac{1}{1+\gamma y^h}$ can be interpreted as the regulatory feedback control employed by nature which captures inhibition of the catalyzing enzyme. Hence, we can derive a control system model for glycolysis as follows

$$\dot{x} = -k_x x + V y^q u \tag{37}$$
$$\dot{y} = -k_y y + (\alpha + \beta) k_x x - \alpha V y^q u, \tag{38}$$

where u is the control input. Our primary motivation behind developing and analyzing such control system models for metabolic pathways is to rigorously prove that the tradeoffs in such models are truly unavoidable and independent of control mechanisms (linear or nonlinear) used to regulate such pathways. The following results assert that essential tradeoffs depend only on autocatalytic structure of the network.

Hard limits on ideal performance. We denote by $y(t; u)$ the output of the system (37)-(38) with respect to control input u. For all possible stabilizing control inputs $u_0 \in L^2([0, \infty))$, i.e., $\int_0^\infty |u_0(t)|^2 dt < \infty$, we can prove that there is a hard limit on the best achievable ideal performance of the system which is characterized as the following inequality

$$\int_0^\infty (y(t; u_0) - \bar{y})^2 dt \geq H(x(0), y(0); \alpha, \beta), \tag{39}$$

in which

$$H(x(0), y(0); \alpha, \beta) = \frac{\alpha^3 \beta k_x}{(\alpha k_y + \beta k_x)^2} \left(z(0) - z^* \right)^2, \tag{40}$$

and $z(0) = x(0) + \frac{1}{\alpha} y(0)$ and $\bar{z} = \bar{x} + \frac{1}{\alpha} \bar{y}$. We define rate of profit ρ as the ratio of β (net production of ATP molecules) to α (number of ATP molecules invested in the pathway), i.e., $\rho = \frac{\beta}{\alpha}$. The hard limit function (40) can be rewritten as

$$H(x(0), y(0); \beta, \rho) = \frac{\beta^2 k_x}{\rho (k_y + \rho k_x)^2} \left(z(0) - z^* \right)^2 \tag{41}$$

Fundamental tradeoff between fragility and net production. When we keep the rate of profit ρ fixed, a fundamental tradeoff between net production of ATP molecules and transient behavior of the system emerges as follows: increasing β can result in undesirable transient behavior (e.g., large–magnitude oscillation in the output of the system) and can increase fragility of the network to small disturbances. For instance, if the level of ATP drops below some threshold, there will not be sufficient supply of ATP for different pathways in the cell and that can result to cell death.

Elimination of hard limits by rewiring the network. We consider a modified model of glycolysis pathway where autocatalysis and regulating control feedbacks are applied on different biochemical reactions. We assume that the enzyme PFK is not regulated. The main motivation for deriving this model is that ATP also inhibits PK activity (the second biochemical reaction). This is a simple abstract model that shows how to eliminate hard limits in autocatalytic networks. The set of ordinary differential equations governing the changes in concentrations x_1 and y is given by

$$\dot{x} = -k_x x \, u + V y^q \tag{42}$$
$$\dot{y} = -k_y y + (\alpha + \beta) k_x x \, u - \alpha V y^q. \tag{43}$$

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where u is the control input. We assume that the desired equilibrium point is still $(\bar{x}, \bar{y})$ in order for a fair comparison between the two models and their corresponding hard limits. We can prove that there is no hard limit on the L^2-norm of the output production, i.e.,

$$\int_0^\infty \left(y(t; u_0) - \bar{y}\right)^2 dt = \mathcal{O}(\epsilon) \tag{44}$$

for all stabilizing state feedback control laws u_0 for system (42)-(43) with $u_0 \in L^2([0, \infty))$. The notation $\mathcal{O}(\epsilon)$ (the big O of ϵ) comes from solving the corresponding cheap optimal control problem. The inequality (44) implies that the value of integral can be made arbitrarily small using cheap optimal controllers [34] and that the transient stability and performance of the system can be improved arbitrarily with no hard limit.

6.3 Synchronization and Collision Avoidance in Nonlinear Dynamical Networks of Agents

In [14], we introduce and discuss two novel second-order consensus networks with state-dependent couplings of Cucker-Smale type. The first scheme models flocking to synchronization over a network of agents where the alignment of the agent's states occurs over a non-trivial limit orbit that is generated by the internal dynamics of each individual agent. The second scheme models the speed alignment of a group of agents which avoid approaching each other closer than a prescribed distance. While seemingly different, both of these systems can be analyzed using the same mathematical methods. We rigorously analyze both examples and reveal their striking similarities. We arrive at sufficient conditions that relate the initial configurations and the systems' parameters that give rise to a collective common behavior. There are several important open problems related to this topic. In [16], we investigate the stability of a class of nonlinear flocking schemes of Cucker-Smale type executed on a finite population of autonomous agents. We use algebraic graph theory tools and derive sufficient conditions for asymptotic convergence for velocity coordination while the flock remains sufficiently connected. In addition, the positions of the agents converge to some predefined relative distances.

7 Centrality Measures in Dynamical Networks

7.1 Notions of Centrality in Consensus Protocols with Structured Uncertainties

In this research thrust, we explored how we can utilize notions of systemic performance measures to identify central nodes and couplings in large-scale networks [22, 30]. This is particularly important for designing robust dynamical networks. In [22, 30], we introduced new insights into the network centrality based not only on the network topology but also on the network dynamics. The focus of this effort was on the class of uncertain linear consensus networks in continuous time, where the network uncertainty is modeled by structured additive Gaussian white noise input on the update dynamics of each agent. The performance of the network is measured by the expected dispersion of its states in steady-state. This measure is equal to the square of the $\mathcal{H}_2$-norm of the network, and it quantifies the extent by which its state is away from the consensus state in steady-state. We show that this performance measure can be explicitly expressed as a function of the Laplacian matrix of the network and the covariance matrix of the noise input. We investigate several structures for the noise input and provide engineering insights on how each uncertainty structure can be relevant in real-world settings. Then, a new centrality index is defined to assess the influence of each agent or link on the network performance. For each noise structure, the value of the centrality index is calculated explicitly, and it is shown that how it depends on the network topology as well as the noise

structure. Our results assert that agents or links can be ranked according to this centrality index and their rank can drastically change from the lowest to the highest, or vice versa, depending on the noise structure.

7.2 Eminence Grise Coalitions: On the Shaping of Public Opinion

in this effort, we characterize another aspect of notion of centrality in time-varying linear dynamical networks. In [6, 7, 22, 29], we have considered an opinion network of multiple individuals with dynamics evolving via a general time-varying continuous time consensus algorithm. In such a network, for an arbitrary fixed initial time, a subset of individuals forms an eminence grise coalition (EGC) if the individuals in that subset are capable of leading the entire network to agreeing on any desired opinion through a cooperative choice of their own initial opinions. In our research, the coalition members are assumed to have access to full profile of the underlying graph of the network as well as the initial opinions of all other individuals. We establish the existence of a minimum size EGC and develop a non-trivial set of upper and lower bounds on that size. Thus, even when the underlying graph does not guarantee convergence to a global or multiple consensus, a generally restricted coalition of individuals can steer public opinion towards a desired consensus provided they can cooperatively adjust their own initial opinions. Moreover, we provide geometric insights into the structure of EGC's. Our results are also extended to the discrete time case where the relation with Decomposition-Separation Theorem is also made explicit.

List of published papers that acknowledge this ONR grant

* List of conference papers appear first. The list of journal papers appear at the end.

[1] M. Siami and N. Motee, "Fundamental limits on robustness measures in networks of interconnected systems," *Proc. of the 52nd IEEE Conference on Decision and Control,* pp. 67-72, Firenze, Italy, Dec. 2013. Published.

[2] N. Motee and Q. Sun, "Measuring sparsity in spatially interconnected systems," *IEEE Conference on Decision and Control,* Firenze, Italy, Dec. 2013. Published.

[3] M. Siami and N. Motee, "Schur-convex robustness measures in dynamical networks," *American Control Conference*, pp. 5198-5203, Portland, OR, June 2014. Published.

[4] N. Motee and Q. Sun, "Measuring sparsity in spatially decaying systems," *American Control Conference*, Portland, OR, June 2014. Published.

[5] M. Siami and N. Motee, "Systemic measures for performance and robustness of large-scale interconnected dynamical networks," *IEEE Conference on Decision and Control*, pp. 5119-5124, Los Angeles, CA, Dec. 2014. Published.

[6] S. Bolouki, R. Malhame, and N. Motee, "A geometric approach towards linear consensus," *The 53rd IEEE Conference on Decision and Control*, pp. pp. 715-720, Los Angeles, CA, Dec. 2014. Published.

[7] S. Bolouki, R. Malhame, M. Siami, and N. Motee, "Eminence grise coalitions in opinion dynamics," *The 52nd Annual Allerton Conf. on Communication, Control, and Computing.* pp. 556-562, October 2014. Published.

[8] M. Siami and N. Motee, "Network Sparsification with Guaranteed Systemic Performance Measures," *In Proc. the 5th IFAC Workshop on Distributed Estimation and Control in Networked Systems, IFAC-PapersOnLine,,* vol. 48, no. 22, pp. 246-251, Philadelphia, PA, 2015. **Best Student Paper Award Winner**

[9] R. Arastoo, M. Bahavarnia, M.V. Kothare and N. Motee, "Output Feedback Controller Sparsification via $\mathcal{H}_2$-Approximation", *In Proc. the 5th IFAC Workshop on Distributed Estimation and Control in Networked Systems, IFAC-PapersOnLine,* vol. 48, no. 22, pp. 112-117, Philadelphia, PA, 2015. Published.

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[14] C. Somarakis, E.Paraskevas, J.S. Baras, and N. Motee, " Synchronization and Collision Avoidance in Nonlinear Flocking Networks of Autonomous Agents," *In Proc. of the 24th Mediterranean Conference on Control and Automation*, Athens, Greece, June 2016. Published.

[15] C. Somarakis, J.S. Baras, and N. Motee, "Consensus and Synchronized Periodicity in Nonlinear Delayed Networks," *In Proc. of the 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS 2016)*, Minneapolis, MN, July 2016. Published.

[16] C. Somarakis and N. Motee, "Nonlinear Flocking with Guaranteed Relative Distances," *The 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2016)*, Monterey, CA, August 2016. Published.

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[18] C. Somarakis, M. Siami and N. Motee, "Interplays Between Systemic Risk and Interconnection Topology in Linear Dynamical Networks," *In Proc. of the 6th IFAC Workshop on Distributed Estimation and Control in Networked Systems, IFAC-PapersOnLine,* vol. 49, no. 22, pp. 333-338, Tokyo, Japan, September 2016. Published.

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